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Finite versions of the Andrews–Gordon identity and Bressoud's identity

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Dedicated to the memory of Professor Basil Gordon

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A B S T R A C T

In this article, we discuss finite versions of Euler's pentagonal number identity, the Rogers-Ramanujan identities and present new proofs of the finite versions of the Andrews-Gordon identity and the Bressoud identity. We also investigate the finite version of Garvan's generalizations of Dyson's rank and discover a new one-variable extension of the Andrews-Gordon identity.

1. Introduction

The famous identity of Euler, which expresses the generating function for the pentagonal numbers in terms of infinite product, states that

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$$\sum_{n=0}^{\infty}(-1)^n q^{(3n^2+n)/2}+\sum_{n=1}^{\infty}(-1)^n q^{(3n^2-n)/2}=\prod_{j=1}^{\infty}(1-q^j). \tag{1.1}$$

Identity (1.1) has an elegant "finite" version due to L.J. Rogers [21] which is given as follows:

Theorem 1.1. *Let n be any positive integer and q be any complex number. Then*

$$\sum_{m=-n}^{n}\frac{(-1)^m q^{m(3m-1)/2}}{(q;q)_{n-m}(q;q)_{n+m}}=\frac{1}{(q;q)_n}, \tag{1.2}$$

where n is a positive integer and $(a;q)_0=1$,

$$(a;q)_n=\prod_{j=1}^{n}(1-aq^{j-1}),$$

$$(a;q)_\infty=\lim_{n\to\infty}\prod_{j=1}^{n}(1-aq^{j-1}).$$

Several finite forms of Euler's pentagonal number theorem can be found in the literature. For example, by replacing n with $3L$ and $3L+1$, respectively, in [22, (16)], two finite forms of Euler's pentagonal number theorem are obtained. Building on this, V.J.W. Guo and J. Zeng [18, (2.11), (2.12)] offered multiple extensions of these two finite forms.

In Section 2, we give a proof of (1.2) followed by some historical remarks about the identity. In Section 3, we use identities discovered by L.J. Rogers, Bailey's transform and identities discussed in Section 2 to derive the Rogers–Ramanujan identities

$$\sum_{j=0}^{\infty}\frac{q^{j^2}}{(q;q)_j}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{1.3}$$

and

$$\sum_{j=0}^{\infty}\frac{q^{j^2+j}}{(q;q)_j}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.4}$$

In Section 4, we derive an analogue of a lemma of D.M. Bressoud used in his elegant proofs of (1.3) and (1.4). The discovery of this analogue is motivated by identities of Rogers discussed in Section 3. We use this analogue to derive new finite versions of (1.4).

In [17], B. Gordon gave a combinatorial generalization of the Rogers–Ramanujan identities for odd moduli. His identity was later converted to a q-series identity by G.E. Andrews [2] as

$$\begin{aligned}&\frac{(q^{k+\nu+2};q^{2k+3})_\infty(q^{k-\nu+1};q^{2k+3})_\infty(q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty} \qquad (1.5)\\&=\sum_{s_k=0}^{\infty}\sum_{0\le s_1\le s_2\le\cdots\le s_{k-1}\le s_k}\frac{q^{U(s_k,s_{k-1},\ldots,s_1)}}{(q;q)_{s_k-s_{k-1}}(q;q)_{s_{k-1}-s_{k-2}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}}\end{aligned}$$

with $0\le\nu\le k$ and where

$$U(s_k,s_{k-1},\ldots,s_1)=\begin{cases}\displaystyle\sum_{j=\nu+1}^{k}s_j^2+\sum_{j=1}^{\nu}(s_j^2+s_j), & \text{if } 0\le\nu<k;\\ \displaystyle\sum_{j=1}^{k}(s_j^2+s_j), & \text{if } \nu=k.\end{cases} \qquad (1.6)$$

Around 1980, Bressoud [11] discovered a generalization of the Rogers–Ramanujan identity for even moduli given by

$$\begin{aligned}&\frac{(q^{k+\nu+1};q^{2k+2})_\infty(q^{k-\nu+1};q^{2k+2})_\infty(q^{2k+2};q^{2k+2})_\infty}{(q;q)_\infty} \qquad (1.7)\\&=\sum_{s_k=0}^{\infty}\sum_{0\le s_1\le s_2\le\cdots\le s_{k-1}\le s_k}\frac{q^{U(s_k,s_{k-1},\ldots,s_1)}}{(q;q)_{s_k-s_{k-1}}(q;q)_{s_{k-1}-s_{k-2}}\cdots(q;q)_{s_2-s_1}(q^2;q^2)_{s_1}}\end{aligned}$$

for $0\le\nu\le k$. Proofs of both (1.5) and (1.7) are also given by P. Paule [19].

In Section 5, we use identities derived from Sections 1 to 4 to prove the following finite versions of the Andrews–Gordon identity and Bressoud's identity.

Theorem 1.2. *Let k be a positive integer and ν be an integer such that $0\le\nu\le k$. Then*

$$\begin{aligned}&\sum_{m=-n}^{n}\frac{(-1)^m q^{((2k+3)m^2+(2\nu+1)m)/2}}{(q;q)_{n-m}(q;q)_{n+m}} \qquad (1.8)\\&=\sum_{s_k=0}^{n}\sum_{0\le s_1\le s_2\le\cdots\le s_{k-1}\le s_k}\frac{q^{U(s_k,s_{k-1},\ldots,s_1)}}{(q;q)_{n-s_k}(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}},\end{aligned}$$

and

$$\begin{aligned}&\sum_{m=-n}^{n}\frac{(-1)^m q^{(k+1)m^2+\nu m}}{(q;q)_{n-m}(q;q)_{n+m}} \qquad (1.9)\\&=\sum_{s_k=0}^{n}\sum_{0\le s_1\le s_2\le\cdots\le s_{k-1}\le s_k}\frac{q^{U(s_k,s_{k-1},\ldots,s_1)}}{(q;q)_{n-s_k}(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q^2;q^2)_{s_1}},\end{aligned}$$

where $U(s_k,s_{k-1},\cdots,s_1)$ is given by (1.6).

We observe that by letting $n \to \infty$ in (1.8) and (1.9), and applying the Jacobi triple product identity

$$(-zq;q)_\infty(-z^{-1};q)_\infty(q;q)_\infty = \sum_{j=-\infty}^{\infty} z^j q^{j(j+1)/2}, \tag{1.10}$$

we arrive at (1.5) and (1.7), respectively. Andrews informed us, through private communication, that one of the cases of (1.8) follows by letting all variables (except a, b, and N) tend to infinity in [3, Theorem 4.1]. One can also deduce (1.8) and (1.9) as special cases of [1, Theorem 3.1]. Identity (1.8) can also be found in [7, Section 3]. Our proofs of (1.8) and (1.9) presented in this article are different from those existing proofs cited above.

The rank of partition was introduced by F. Dyson [13] to provide combinatorial interpretations for Ramanujan's renowned partition congruences modulo 5 and 7. Concurrently, the notion of the crank of partitions, initially conjectured by Dyson and subsequently discovered by Andrews and Garvan [6], offers combinatorial interpretations for all three of Ramanujan's partition congruences modulo 5, 7, and 11. Expanding on this groundwork, Garvan introduced a k-rank generalization of partition ranks and cranks in [14], where setting $k = 1$ yields the Andrews–Garvan crank, and $k = 2$ corresponds to the Dyson rank.

The finite forms for the rank generating function and its variant can be found in [5, (12.2.2) and (12.2.4)]. On page 264 of [5], Andrews and Berndt gave the finite form of the crank generating function

$$\sum_{m=-n}^{n} \frac{(-1)^m q^{(m^2+m)/2}(1-c)}{(q;q)_{n-m}(q;q)_{n+m}(1-cq^m)} = \frac{1}{(cq;q)_n(q/c;q)_n}. \tag{1.11}$$

Motivated by the identities found in [5, (12.2.2) and (12.2.4)] and (1.11), we undertake an exploration to find the finite forms for the generalized k-rank. Our result is as follows.

Theorem 1.3. *Let k and ν be integers with $0 \le \nu \le k$. Then*

$$\begin{aligned}&\sum_{m=-n}^{n} \frac{(-1)^m q^{((2k+1)m^2+(2\nu+1)m)/2}(1-c)}{(q;q)_{n-m}(q;q)_{n+m}(1-cq^m)} \\ &= \sum_{s_k=0}^{n}\sum_{s_{k-1}=0}^{s_k}\cdots\sum_{s_1=0}^{s_2} \frac{q^{U(s_k,s_{k-1},\ldots,s_2,s_1)}V(q)}{(q;q)_{n-s_k}(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(cq;q)_{s_1}(q/c;q)_{s_1+1}},\end{aligned} \tag{1.12}$$

where

$$V(q) = \begin{cases} 1-q^{s_{\nu+1}+1}/c, & \text{if } 0 \le \nu \le k-1; \\ 1-q^{n+1}/c, & \text{if } \nu = k. \end{cases}$$

In Section 6, we prove Theorem 1.3 and give a few immediate consequences of the theorem. One of them demonstrates that the finite form (1.12) leads to a new Andrews–Gordon type identity (6.1), which is a new one-variable generalization of the Andrews–Gordon identity (1.5).

Our proofs of the finite forms of the Andrews-Gordon identities rely on three key steps: iterating the sums (for example, (5.1), (5.2), (6.5), and (6.6)), transitioning between balanced and unbalanced sums (for example, (5.3) and (6.7)), and determining the initial values (for example, (5.6) and (6.10)). While the use of sum iterations is not new and has been employed in previous proofs, a distinguishing feature of our approach is the novel transition between balanced and unbalanced sums. This systematic framework simplifies the derivation of Andrews-Gordon type identities, eliminating the need for ad-hoc techniques or the discovery of Bailey Lattices for each case. By employing this elementary and structured method, our approach offers enhanced clarity on the inherent structures of these identities.

2. Euler's pentagonal number identity

In this section, we give a proof of (1.2). We begin with an identity connecting two finite sums.

Lemma 2.1. *Let*

$$\gamma_n = \sum_{r=-n}^{n} (-1)^r q^{r(3r+1)/2} \begin{bmatrix} 2n \\ n-r \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_r (q;q)_{n-r}} & \text{if } 0 \le r \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then

$$\sum_{r=-(n+1)}^{n-1} (-1)^r q^{3r(r+1)/2} \begin{bmatrix} 2n \\ n-r-1 \end{bmatrix} = -q^n \gamma_n. \tag{2.2}$$

Proof. By (2.1), we observe that to prove (2.2), it suffices to show that

$$\sum_{r=-\infty}^{\infty} (-1)^r q^{3r(r+1)/2} \begin{bmatrix} 2n \\ n-r-1 \end{bmatrix} = -q^n \sum_{r=-\infty}^{\infty} (-1)^r q^{r(3r+1)/2} \begin{bmatrix} 2n \\ n-r \end{bmatrix}.$$

It is known that

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r \end{bmatrix} + q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} \tag{2.3}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}. \tag{2.4}$$

Using (2.4), we deduce that

$$\begin{bmatrix} 2n+1 \\ n-r \end{bmatrix} = q^{n-r} \begin{bmatrix} 2n \\ n-r \end{bmatrix} + \begin{bmatrix} 2n \\ n-r-1 \end{bmatrix}. \tag{2.5}$$

Next, observe that

$$\sum_{r=-\infty}^{\infty} (-1)^r q^{3r(r+1)/2} \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix} = \sum_{r'=-\infty}^{\infty} (-1)^{r'} q^{3r'(r'-1)/2} \begin{bmatrix} 2n+1 \\ n+r' \end{bmatrix},$$

where we have let $r' = -r$. Setting $s = r' - 1$, we conclude that

$$\begin{aligned} \sum_{r=-\infty}^{\infty} (-1)^r q^{3r(r+1)/2} \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix} &= \sum_{s=-\infty}^{\infty} (-1)^{s+1} q^{3s(s+1)/2} \begin{bmatrix} 2n+1 \\ n+s+1 \end{bmatrix} \\ &= \sum_{s=-\infty}^{\infty} (-1)^{s} q^{3s(s+1)/2} \begin{bmatrix} 2n+1 \\ n-s \end{bmatrix}. \end{aligned}$$

Therefore,

$$\sum_{r=-\infty}^{\infty} (-1)^r q^{3r(r+1)/2} \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix} = 0. \tag{2.6}$$

Identity (2.6) is due to L.J. Rogers [4, (3.12)].

Substituting (2.5) into the (2.6), we deduce that

$$q^n \sum_{r=-\infty}^{\infty} (-1)^r q^{r(3r+1)/2} \begin{bmatrix} 2n \\ n-r \end{bmatrix} + \sum_{r=-\infty}^{\infty} (-1)^r q^{3r(r+1)/2} \begin{bmatrix} 2n \\ n-r-1 \end{bmatrix} = 0$$

and the proof of (2.2) is complete. □

We are now ready to prove (1.2).

Theorem 2.1. *For any positive integer n,*

$$\gamma_n = \frac{(q;q)_{2n}}{(q;q)_n},$$

or

$$\sum_{r=-n}^{n}(-1)^r q^{r(3r+1)/2}\begin{bmatrix}2n\\ n-r\end{bmatrix}=\frac{(q;q)_{2n}}{(q;q)_n}. \tag{2.7}$$

Note that (2.7) is the same as (1.2) but (1.2) is the identity that we wish to publicize.

Proof. From (2.3) and (2.4), we deduce that

$$\begin{bmatrix}2n\\ n-r\end{bmatrix}=q^{2n-2r}\begin{bmatrix}2n-2\\ n-r\end{bmatrix}+q^{n-r}\begin{bmatrix}2n-2\\ n-r-1\end{bmatrix}+\begin{bmatrix}2n-2\\ n-r-1\end{bmatrix}+q^{n+r}\begin{bmatrix}2n-2\\ n-r-2\end{bmatrix}.$$

Therefore,

$$\begin{aligned}\gamma_n=&\sum_{r=-\infty}^{\infty}(-1)^r q^{r(3r+1)/2}q^{2n-2r}\begin{bmatrix}2n-2\\ n-r\end{bmatrix}+\sum_{r=-\infty}^{\infty}(-1)^r q^{r(3r+1)/2}q^{n-r}\begin{bmatrix}2n-2\\ n-r-1\end{bmatrix}\\ &+\sum_{r=-\infty}^{\infty}(-1)^r q^{r(3r+1)/2}\begin{bmatrix}2n-2\\ n-r-1\end{bmatrix}+\sum_{r=-\infty}^{\infty}(-1)^r q^{r(3r+1)/2}q^{n+r}\begin{bmatrix}2n-2\\ n-r-2\end{bmatrix}.\end{aligned} \tag{2.8}$$

The second and third series of (2.8) are $q^n\gamma_{n-1}$ and γ_{n-1}, respectively. The first series is

$$\begin{aligned}\sum_{r=-\infty}^{\infty}(-1)^r q^{r(3r+1)/2}q^{2n-2r}\begin{bmatrix}2n-2\\ n-r\end{bmatrix}&=q^{2n}\sum_{r=-\infty}^{\infty}(-1)^r q^{3r(r-1)/2}\begin{bmatrix}2n-2\\ n-r\end{bmatrix}\\ &=q^{2n}\sum_{r=-\infty}^{\infty}(-1)^r q^{3r(r+1)/2}\begin{bmatrix}2n-2\\ n+r\end{bmatrix}\\ &=q^{2n}\sum_{r=-\infty}^{\infty}(-1)^r q^{3r(r+1)/2}\begin{bmatrix}2n-2\\ n-r-2\end{bmatrix}\\ &=-q^{3n-1}\gamma_{n-1},\end{aligned}$$

where the last equality follows from (2.2). Similarly, the fourth series in (2.8) is $-q^{2n-1}\gamma_{n-1}$. Therefore,

$$\gamma_n=(1+q^n-q^{2n-1}-q^{3n-1})\gamma_{n-1}=(1+q^n)(1-q^{2n-1})\gamma_{n-1} \tag{2.9}$$

and we deduce that

$$\gamma_n=\frac{(q;q)_{2n}}{(q;q)_n}.\quad\square$$

Remark 2.1.

1. The recurrence (2.9) satisfied by γ_n can be proved by producing a qWZ-certificate for the identity. For more details, see Paule's article [20]. The above proof may be viewed as a human discovery of a qWZ-certificate (and hence a proof of (1.2) obtained by a computer).
2. The proof of (2.7) (or equivalently (1.2)) is similar to the proof given by Rogers and presented by Andrews [4, Chapter 3]. Andrews defined

$$\beta_{2n} = q^{n^2+n} \sum_{r=-\infty}^{\infty} (-1)^r q^{r(3r+1)/2} \begin{bmatrix} 2n \\ n-r \end{bmatrix}$$

and

$$\beta_{2n+1} = q^{(n+1)^2} \sum_{r=-\infty}^{\infty} (-1)^r q^{r(3r+1)/2} \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix}$$

and deduce that

$$\beta_{2n+1} - q^{n+1}\beta_{2n} = -q^{3n+2}\beta_{2n} \quad \text{and} \quad \beta_{2n} - q^n\beta_{2n-1} = q^{2n}\beta_{2n-1}$$

and conclude that [4, (3.17), (3.18)]

$$\beta_{2n} = q^{n(n+1)}(q^{n+1};q)_n = q^{n(n+1)}\frac{(q;q)_{2n}}{(q;q)_n} \quad \text{and} \quad \beta_{2n-1} = q^{n^2}(q^n;q)_n.$$

Note that the expression for β_{2n} is (2.7).
3. Identity (1.2) has been rediscovered several times. For example, it is proved in [19, (10)] using the identity of Bressoud (with $a = 3/2$ and $x = -q^{1/2}$) [10, (18)]

$$\sum_{j=-n}^{n} \frac{x^j q^{aj^2}}{(q;q)_{n-j}(q;q)_{n+j}} = \sum_{s=0}^{n} \frac{q^{s^2}}{(q;q)_{n-s}} \sum_{j=-s}^{s} \frac{x^j q^{(a-1)j^2}}{(q;q)_{s-j}(q;q)_{s+j}} \tag{2.10}$$

and the fact that

$$\sum_{j=-s}^{s} \frac{(-1)^j q^{j(j+1)/2}}{(q;q)_{s-j}(q;q)_{s+j}} = \begin{cases} 1 & \text{if } s=0, \\ 0 & \text{otherwise.} \end{cases}$$

4. Identity (1.2) also appeared in an article by A. Berkovich and S.O. Warnaar [8, (1.4)].
5. If we replace q by q^{-1} in (1.2), we obtain the identity

$$(-1)^n \frac{q^{-n(n+1)/2}}{(q;q)_n} = \sum_{m=-n}^{n} (-1)^m \frac{q^{-m(m+1)/2}}{(q;q)_{n-m}(q;q)_{n+m}}.$$

This identity follows by letting $b, c \to 0$ in Jackson's terminating q-analogue of Dixon's sum [15, p. 355, (II.15)]

$$ {}_3\varphi_2\left(\begin{matrix} q^{-2n}, & b, & c \\ q^{1-2n}/b, & q^{1-2n}/c \end{matrix}; q; \frac{q^{2-n}}{bc}\right) = \frac{(b;q)_n(c;q)_n(bc;q)_{2n}(q;q)_{2n}}{(b;q)_{2n}(c;q)_{2n}(bc;q)_n(q;q)_n} $$

where

$$ {}_3\varphi_2\left(\begin{matrix} a, & b, & c \\ d, & e \end{matrix}; q; z\right) = \sum_{j=0}^{\infty} \frac{(a;q)_j(b;q)_j(c;q)_j}{(d;q)_j(e;q)_j(q;q)_j} z^j. $$

This gives another proof of (1.2). One can give several different proofs of (1.2) if one uses different identities involving the basic hypergeometric series.

3. Jacobi's triple product identity, Bailey's transform, (1.2) and the Rogers–Ramanujan identities

In Remark 2.1.3, we mentioned the connection of (1.2) with Bressoud's elegant proofs of the Rogers–Ramanujan identities (1.3) and (1.4).

In this section, we show how to derive (1.3) and (1.4) using (1.2).

In [4, (3.3)-(3.5)], G.E. Andrews recorded the identity

$$ (q;q)_\infty \left(1 + \sum_{j=0}^{\infty} B_n(\theta) \frac{q^{-n}}{(q;q)_n}\right) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos n\theta, \tag{3.1} $$

where

$$ B_{2n}(\theta) = q^{n^2+n}\left(\frac{(q;q)_{2n}}{(q;q)_n(q;q)_n} + 2\sum_{r=1}^{n} q^{r^2} \frac{(q;q)_{2n}}{(q;q)_{n-r}(q;q)_{n+r}} \cos 2r\theta\right) $$

and

$$ B_{2n+1}(\theta) = 2q^{(n+1)^2}\left(\sum_{r=0}^{n} q^{r(r+1)} \frac{(q;q)_{2n+1}}{(q;q)_{n-r}(q;q)_{n+r+1}} \cos(2r+1)\theta\right). $$

Comparing the coefficients of $\cos n\theta$ on both sides of (3.1), we deduce the following interesting identities:

Lemma 3.1. *Let n be a positive integer and $|q| < 1$. Then*

$$ q^{n^2} = (q;q)_\infty \sum_{s\ge n} \frac{q^{s^2}}{(q;q)_{s-n}(q;q)_{s+n}}, \tag{3.2} $$

and

$$q^{n^2+n} = (q;q)_\infty \sum_{s\geq n} \frac{q^{s^2+s}}{(q;q)_{s-n}(q;q)_{s+n+1}}. \tag{3.3}$$

Identities (3.2) and (3.3) are not explicitly stated in [4, p. 22].

We now study the following two lemmas, which are special cases of Bailey's transform.

Theorem 3.1.

(a) *If*

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_n}{(q;q)_{n-j}(q;q)_{n+j}},$$

then

$$\sum_{n=0}^{\infty} q^{n^2}\beta_n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{n^2}\alpha_n. \tag{3.4}$$

(b) *If*

$$\beta_n^* = \sum_{j=0}^{n} \frac{\alpha_n^*}{(q;q)_{n-j}(q;q)_{n+j+1}}, \tag{3.5}$$

then

$$\sum_{n=0}^{\infty} q^{n^2+n}\beta_n^* = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{n^2+n}\alpha_n^*. \tag{3.6}$$

Proof. The proof is simply an application of interchanging the order of summation of two variable. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} q^{n^2}\beta_n &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} q^{n^2} \frac{\alpha_r}{(q;q)_{n-r}(q;q)_{n+r}} \\
&= \sum_{r=0}^{\infty} \alpha_r \sum_{n\geq r} \frac{q^{n^2}}{(q;q)_{n-r}(q;q)_{n+r}} \\
&= \frac{1}{(q;q)_\infty} \sum_{r=0}^{\infty} q^{r^2}\alpha_r,
\end{aligned}$$

where we have used (3.2) in the last equality. This completes the proof of (3.4). The proof of (3.6) is similar. □

The sequences $\{\alpha_n | n \geq 0\}$ and $\{\beta_n | n \geq 0\}$ are known as "Bailey pairs".

Identity (3.4) is the case $a = 1$ of [4, (3.33)], which was proved using Bailey's Lemma. We emphasize that, in our context, Bailey's Lemma is not needed for the proofs of (3.4) and (3.6) because of (3.2) and (3.3).

We are now ready to derive (1.3) and (1.4) from (1.2), or more precisely, from (2.2). First, we note that

$$\frac{1}{(q;q)_n} = \sum_{r=-n}^{n} \frac{(-1)^r q^{r(3r-1)/2}}{(q;q)_{n-r}(q;q)_{n+r}} = \frac{1}{(q;q)_n^2} + \sum_{r=1}^{n} \frac{(-1)^r q^{r(3r-1)/2}(1+q^r)}{(q;q)_{n-r}(q;q)_{n+r}}.$$

Therefore, we may let

$$\beta_n = \frac{1}{(q;q)_n}$$

and $\alpha_0 = 1$ and for $n \geq 1$,

$$\alpha_n = (-1)^n q^{n(3n-1)/2}(1+q^n)$$

and substitute them into (3.4). Upon doing so, we deduce that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q)_\infty}\left(1 + \sum_{n=1}^{\infty}(-1)^n q^{n(5n-1)/2}(1+q^n)\right) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

which gives (1.3).

Similarly, note that from (2.2),

$$\frac{q^n}{(q;q)_n} = \sum_{r=-n}^{n} \frac{(-1)^r q^{3r(r-1)/2}}{(q;q)_{n-r}(q;q)_{n+r}} = \frac{1}{(q;q)_n^2} + \sum_{r=1}^{n} \frac{(-1)^r q^{3r(r-1)/2}(1+q^{3r})}{(q;q)_{n-r}(q;q)_{n+r}}.$$

By applying (3.4) with

$$\beta_n = \frac{q^n}{(q;q)_n}$$

and $\alpha_0 = 1$ and for $n \geq 1$,

$$\alpha_n = (-1)^n q^{3n(n-1)/2}(1+q^{3n}),$$

we deduce that

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q;q)_\infty}\left(1 + \sum_{n=1}^{\infty}(-1)^n q^{n(5n-3)/2}(1+q^{3n})\right) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

which is (1.4).

Remark 3.1. The simplest proofs of (1.3) and (1.4) are perhaps those given by Bressoud [10]. In fact, using (1.2) and Bressoud's (2.10) with $a = 5/2$ implies the finite form of (1.3), namely,

$$\sum_{j=-n}^{n}(-1)^j\frac{q^{(5j^2+j)/2}}{(q;q)_{n-j}(q;q)_{n+j}} = \sum_{s=0}^{n}\frac{q^{s^2}}{(q;q)_{n-s}(q;q)_s}. \tag{3.7}$$

Similarly, using (2.2) and $a = 5/2$ in (2.10), we deduce that

$$\begin{aligned}\sum_{j=-n}^{n}(-1)^j\frac{q^{(5j^2-3j)/2}}{(q;q)_{n-j}(q;q)_{n+j}} &= \sum_{j=-n}^{n}(-1)^j\frac{q^{(5j^2+3j)/2}}{(q;q)_{n-j}(q;q)_{n+j}} \\ &= \sum_{s=0}^{n}\frac{q^{s^2+s}}{(q;q)_{n-s}(q;q)_s}.\end{aligned} \tag{3.8}$$

Identity (3.8) is a finite form of (1.4). It can also be derived from Bressoud's identity [10, p. 237, after (10)].

Identity (3.7) suggests that if $\alpha_0 = 1$ and for $n \geq 1$,

$$\alpha_n = (-1)^n q^{n(5n-1)/2}(1+q^n) \quad \text{and} \quad \beta_n = \sum_{s=0}^{n}\frac{q^{s^2}}{(q;q)_{n-s}(q;q)_s},$$

then (3.4) implies that

$$\sum_{j=0}^{\infty}\frac{q^{j^2}}{(q;q)_j}\sum_{s=0}^{j}\begin{bmatrix} j \\ s \end{bmatrix} q^{s^2} = \frac{(q^3;q^7)_\infty(q^4;q^7)_\infty(q^7;q^7)_\infty}{(q;q)_\infty}. \tag{3.9}$$

Similarly, using (3.8) and (3.4), we deduce that

$$\sum_{j=0}^{\infty}\frac{q^{j^2}}{(q;q)_j}\sum_{s=0}^{j}\begin{bmatrix} j \\ s \end{bmatrix} q^{s^2+s} = \frac{(q^2;q^7)_\infty(q^5;q^7)_\infty(q^7;q^7)_\infty}{(q;q)_\infty}. \tag{3.10}$$

The above identities can also be obtained from Bressoud's identity (2.10).

4. An analogue of Bressoud's lemma and its consequences

In the previous section, we have seen that (1.2) and (2.2), together with (3.4) imply (1.3) and (1.4). We notice that (3.6) is not involved in the discussion. An attempt to understand (3.6) leads us to the following analogue of Bressoud's identity (2.10) [10, (18)]:

Lemma 4.1. *Let x be a non-zero complex number and q be any complex number. Then*

$$\sum_{j=-n-1}^{n} \frac{x^j q^{a(j^2+j)}}{(q;q)_{n-j}(q;q)_{n+j+1}} = \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}} \sum_{m=-s-1}^{s} \frac{x^m q^{(a-1)(m^2+m)}}{(q;q)_{s-m}(q;q)_{s+m+1}}. \tag{4.1}$$

We are led to Lemma 4.1 that involves $(q;q)_{n+j+1}$ instead of $(q;q)_{n+j}$ in the denominator of (4.1) because we are looking for examples of β_n^* and α_n^* in (3.5). A generalization of (4.1) can be found in [19, (R2)].

Proof of Lemma 4.1. First, by setting $m' = -m$ followed by letting $\ell = m' - 1$ be the new summation index, we find that

$$\begin{aligned}\sum_{m=-n-1}^{-1} \frac{x^m q^{a(m^2+m)}}{(q;q)_{n-m}(q;q)_{n+m+1}} &= \sum_{m'=1}^{n+1} \frac{x^{-m'} q^{a(m'^2-m')}}{(q;q)_{n+m'}(q;q)_{n-m'+1}} \\ &= \sum_{\ell=0}^{n} \frac{x^{-\ell-1} q^{a(\ell^2+\ell)}}{(q;q)_{n+\ell+1}(q;q)_{n-\ell}}.\end{aligned}$$

Therefore, we may rewrite the left hand side of (4.1) as

$$\sum_{j=-n-1}^{n} \frac{x^j q^{a(j^2+j)}}{(q;q)_{n-j}(q;q)_{n+j+1}} = \sum_{j=0}^{n} \frac{(x^j + x^{-j-1}) q^{a(j^2+j)}}{(q;q)_{n-j}(q;q)_{n+j+1}}. \tag{4.2}$$

Next, we recall from [10, Lemma 1] that

$$\frac{1}{(xq;q)_k} = \sum_{j=0}^{k} \frac{x^j q^{j^2}}{(xq;q)_j} \begin{bmatrix} k \\ j \end{bmatrix}. \tag{4.3}$$

For integers $n \geq m \geq 0$, let $k = n - m$ and $x = q^{2m+1}$ in (4.3), we deduce that

$$\frac{1}{(q;q)_{n+m+1}} = \sum_{j=0}^{n-m} \frac{q^{2mj+j^2+j}}{(q;q)_{2m+1+j}} \frac{(q;q)_{n-m}}{(q;q)_j (q;q)_{n-m-j}}. \tag{4.4}$$

Using (4.2) and (4.4), we deduce that

$$\begin{aligned}&\sum_{m=-n-1}^{n} \frac{x^m q^{a(m^2+m)}}{(q;q)_{n-m}(q;q)_{n+m+1}} \\ &= \sum_{m=0}^{n} \frac{(x^m + x^{-m-1}) q^{a(m^2+m)}}{(q;q)_{n-m}} \sum_{j=0}^{n-m} \frac{q^{2mj+j^2+j}}{(q;q)_{2m+1+j}} \frac{(q;q)_{n-m}}{(q;q)_j (q;q)_{n-m-j}} \\ &= \sum_{m=0}^{n} \sum_{j=0}^{n-m} \frac{(x^m + x^{-m-1}) q^{(a-1)m^2} q^{(m+j)^2} q^j q^{am}}{(q;q)_j (q;q)_{n-m-j} (q;q)_{2m+1+j}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} \sum_{s=m}^{n} \frac{(x^m + x^{-m-1}) q^{(a-1)(m^2+m)} q^{s^2+s}}{(q;q)_{s-m}(q;q)_{m+s+1}(q;q)_{n-s}} \\
&= \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}} \sum_{m=0}^{s} \frac{(x^m + x^{-m-1}) q^{(a-1)(m^2+m)}}{(q;q)_{s-m}(q;q)_{m+s+1}} \\
&= \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}} \sum_{m=-s-1}^{s} \frac{x^m q^{(a-1)(m^2+m)}}{(q;q)_{s-m}(q;q)_{m+s+1}},
\end{aligned}
$$

where in the last equality, we applied (4.2) on the inner sum. □

As in Bressoud's article [10], in order to prove identities such as (1.2) and (1.3), we need to determine an identity to "kick start" the process. This identity is given by

Lemma 4.2. *Let q be any complex number. Then*

$$
\sum_{m=-n-1}^{n} \frac{(-1)^m q^{m(m-1)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}
$$

Identity (4.5) can be derived from the following Lemma by using the substitution $z = -1$.

Lemma 4.3. *Let x be a non-zero complex number and q be any complex number. Then*

$$
(-z;q)_N(-qz^{-1};q)_{N+1} = \sum_{j=-N-1}^{N} z^j q^{(j^2-j)/2} \begin{bmatrix} 2N+1 \\ N-j \end{bmatrix}. \tag{4.6}
$$

Proof. The finite q-binomial [9, (3.3.3)] states that

$$
(-yq;q)_n = \sum_{j=0}^{n} y^j q^{j(j+1)/2} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{4.7}
$$

Replace n by $2N+1$ and y by $q^{-N-1}z$ in (4.7), we obtain (4.6) after simplifications. □

In most articles or books in the literature, the identity (see for example [9, (3.5.5)] or [10, (6)])

$$
(-zq;q)_N(-z^{-1};q)_N = \sum_{j=-N}^{N} z^j q^{j(j+1)/2} \begin{bmatrix} 2N \\ N-j \end{bmatrix} \tag{4.8}
$$

is proved after proving (4.7). The Jacobi triple product identity (1.10) is then deduced from (4.8) by letting N tend to ∞. We observe that (4.6), which corresponds to the "odd" case, is lesser known. It is immediate that (1.10) also follows from (4.6) by letting N tends to ∞, giving us another proof of (1.10).

Corollary 4.1. *Let q be any complex number. Then*
(a)

$$\sum_{m=-n-1}^{n} \frac{(-1)^m q^{(3m^2+m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \frac{1}{(q;q)_n} \tag{4.9}$$

and
(b)

$$\sum_{m=-n-1}^{n} \frac{(-1)^m q^{(5m^2+3m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}(q;q)_s}. \tag{4.10}$$

Proof. (a) Let $a = 3/2$ and $x = -q^{-1}$ in (4.1) and observe that

$$\begin{aligned}\sum_{j=-n-1}^{n} \frac{(-1)^j q^{(3j^2+j)/2}}{(q;q)_{n-j}(q;q)_{n+j+1}} &= \sum_{m=0}^{n} \frac{q^{m^2+m}}{(q;q)_{n-m}} \sum_{r=-m-1}^{m} \frac{(-1)^r q^{(r^2-r)/2}}{(q;q)_{m-r}(q;q)_{m+r+1}} \\ &= \frac{1}{(q;q)_n},\end{aligned}$$

where the last equality follows from (4.5) and the proof of (4.9) is complete. We observe that (4.9) is another finite version of Euler's identity (1.1).

(b) The proof of (4.10) follows from (4.1) with $a = 5/2$ and $x = -q^{-1}$ and (4.9). We note that (4.10) is another finite version of (1.4). □

We wish to highlight that by combining (4.9) and (1.2), we have the identities

$$\sum_{m=-n-1}^{n} \frac{(-1)^m q^{(3m^2+m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \sum_{m=-n}^{n} \frac{(-1)^m q^{(3m^2+m)/2}}{(q;q)_{n-m}(q;q)_{n+m}} = \frac{1}{(q;q)_n}. \tag{4.11}$$

Similarly, by combining (4.10) and (3.8), we deduce the identities

$$\sum_{m=-n-1}^{n} \frac{(-1)^m q^{(5m^2+3m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \sum_{m=-n}^{n} \frac{(-1)^m q^{(5m^2+3m)/2}}{(q;q)_{n-m}(q;q)_{n+m}} = \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}(q;q)_s}. \tag{4.12}$$

Next, if we let $a = 7/2$ and $x = 1/q$ in (4.1), and substituting (4.10) on the right hand side, we deduce that

$$\begin{aligned}\sum_{j=-n-1}^{n} \frac{q^{(7j^2+5j)/2}}{(q;q)_{n-j}(q;q)_{n+j+1}} &= \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}} \sum_{m=-s-1}^{s} \frac{q^{(5m^2+3m)/2}}{(q;q)_{s-m}(q;q)_{s+m+1}} \\ &= \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}} \sum_{m=0}^{s} \frac{q^{m^2+m}}{(q;q)_{m-s}(q;q)_m}.\end{aligned}$$

Letting n tend to ∞ in the above identity, we deduce, after an application of (1.10), the following septic identity which is a companion of (3.9) and (3.10):

$$\sum_{s=0}^{\infty} \frac{q^{s^2+s}}{(q)_s} \sum_{m=0}^{s} \begin{bmatrix} s \\ m \end{bmatrix} q^{m^2+m} = \frac{(q;q^7)_\infty (q^6;q^7)_\infty (q^7;q^7)_\infty}{(q;q)_\infty}. \tag{4.13}$$

Identity (4.13) can also be proved by observing that we may obtain a Bailey pair satisfying (3.5) from (4.10). By applying the Bailey pair obtained on (3.6) gives another proof of (4.13).

Identities (4.11) and (4.12) are special cases of the following important identity.

Theorem 4.1. *Let a be a positive integer and q be a complex number. Then*

$$\sum_{m=-n-1}^{n} \frac{(-1)^m q^{(am^2+(a-2)m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \sum_{m=-n}^{n} \frac{(-1)^m q^{(am^2+(a-2)m)/2}}{(q;q)_{n-m}(q;q)_{n+m}}. \tag{4.14}$$

Proof. We shall verify (4.14). First, we rewrite the right hand side of (4.14) as

$$\begin{aligned} &\sum_{m=-n}^{n} \frac{(-1)^m q^{(am^2+(a-2)m)/2}}{(q;q)_{n-m}(q;q)_{n+m}} \frac{1-q^{n+m+1}}{1-q^{n+m+1}} \\ &\quad = \sum_{m=-n}^{n} \frac{(-1)^m q^{(am^2+(a-2)m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} - \sum_{m=-n}^{n} \frac{(-1)^m q^{(am^2+(a-2)m)/2+n+m+1}}{(q;q)_{n-m}(q;q)_{n+m+1}}. \end{aligned}$$

Comparing with the left hand side of (4.14), we conclude that to prove (4.14), it suffices to show that

$$\sum_{m=-n}^{n} \frac{(-1)^m q^{(am^2+am)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \frac{(-1)^n q^{(an^2+an)/2}}{(q;q)_{2n+1}}. \tag{4.15}$$

Now,

$$\begin{aligned} \sum_{m=-n}^{n} \frac{(-1)^m q^{(am^2+am)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} &= \frac{(-1)^n q^{(an^2+an)/2}}{(q;q)_{2n+1}} \\ &\quad + \sum_{m=-n}^{-1} \frac{(-1)^m q^{(am^2+am)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} + \sum_{m=0}^{n-1} \frac{(-1)^m q^{(am^2+am)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}}. \end{aligned}$$

Therefore, it suffices to show that

$$\sum_{m=-n}^{-1} \frac{(-1)^m q^{(am^2+am)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}} = -\sum_{m=0}^{n-1} \frac{(-1)^m q^{(am^2+am)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}},$$

which is true by replacing m on the left hand side of the expression by $-m'$, followed by replacing m' by $m''+1$. We observe here that (2.6) is the case $a=3$ of (4.15) and as a result, we have another proof of (2.6). □

We remark that by uniqueness theorem in complex analysis, we may deduce from (4.14) the following identity:

Corollary 4.2. *Let x and q be complex numbers and n be a positive integer. Then*

$$\sum_{m=-n-1}^{n} \frac{(-1)^m x^{m(m+1)} q^{-m}}{(q;q)_{n-m}(q;q)_{n+m+1}} = \sum_{m=-n}^{n} \frac{(-1)^m x^{m(m+1)} q^{-m}}{(q;q)_{n-m}(q;q)_{n+m}}.$$

5. The Andrews–Gordon identity and Bressoud's identity

We are now ready to give the proof of Theorem 1.2.

Proof. The identities required to prove (1.8) are (2.10), (4.1) and (4.14). We introduce some functions to simplify the presentation of our proof. Let

$$\xi(\mu,\nu,j) = \sum_{m=-j}^{j} \frac{(-1)^m q^{((2\mu+3)m^2+(2\nu+1)m)/2}}{(q;q)_{j-m}(q;q)_{j+m}}$$

and

$$\chi(\mu,j) = \sum_{m=-j-1}^{j} \frac{(-1)^m q^{((2\mu+3)(m^2+m)-2m)/2}}{(q;q)_{j-m}(q;q)_{j+m+1}}.$$

With the above notations, we may deduce from (2.10) and (4.1) that

$$\xi(\mu,\nu,n) = \sum_{m=0}^{n} \frac{q^{m^2}}{(q;q)_{n-m}} \xi(\mu-1,\nu,m) \tag{5.1}$$

and

$$\chi(\mu,n) = \sum_{m=0}^{n} \frac{q^{m^2+m}}{(q;q)_{n-m}} \chi(\mu-1,m). \tag{5.2}$$

Using the above notations, we may rewrite (4.14) as

$$\xi(\nu,\nu,n) = \chi(\nu,n). \tag{5.3}$$

Suppose $k>\nu$. Then applying (5.1), we arrive at

$$\xi(k,\nu,n)=\sum_{s_k=0}^{n}\frac{q^{s_k^2}}{(q;q)_{n-s_k}}\sum_{s_{k-1}=0}^{s_k}\frac{q^{s_{k-1}^2}}{(q;q)_{n-s_{k-1}}}\cdots\sum_{s_{\nu+1}=0}^{s_{\nu+2}}\frac{q^{s_{\nu+1}^2}}{(q;q)_{n-s_{\nu+1}}}\xi(\nu,\nu,s_\nu). \tag{5.4}$$

Once we encounter $\xi(\nu,\nu,s_\nu)$, we use (5.3) to replace $\xi(\nu,\nu,s_\nu)$ with $\chi(\nu,s_\nu)$. We then apply (5.2) to simplify the right hand side of (5.4) as

$$\begin{aligned}\xi(k,\nu,n)=&\sum_{s_k=0}^{n}\frac{q^{s_k^2}}{(q;q)_{n-s_k}}\sum_{s_{k-1}=0}^{s_k}\frac{q^{s_{k-1}^2}}{(q;q)_{n-s_{k-1}}}\cdots\sum_{s_{\nu+1}=0}^{s_{\nu+2}}\frac{q^{s_{\nu+1}^2}}{(q;q)_{n-s_{\nu+1}}}\\&\sum_{s_\nu=0}^{s_{\nu+1}}\frac{q^{s_\nu^2+s_\nu}}{(q;q)_{s_{\nu+1}-s_\nu}}\cdots\sum_{s_1=0}^{s_2}\frac{q^{s_1^2+s_1}}{(q;q)_{s_2-s_1}}\chi(0,s_1).\end{aligned} \tag{5.5}$$

Using (4.9), we conclude that

$$\chi(0,s_1)=\sum_{m=-s_1-1}^{s_1}\frac{(-1)^{s_1}q^{(3(s_1^2+s_1)-2s_1)/2}}{(q;q)_{s_1-m}(q;q)_{s_1+m+1}}=\frac{1}{(q;q)_{s_1}}. \tag{5.6}$$

Substituting (5.6) into (5.5) completes the proof of (1.8).

The proof of (1.9) is similar to that of (1.8) except for the final step. We present the relevant steps. Let

$$\xi^*(\mu,\nu,j)=\sum_{m=-j}^{j}\frac{(-1)^m q^{(\mu+1)m^2+\nu m}}{(q;q)_{j-m}(q;q)_{j+m}}$$

and

$$\chi^*(\mu,j)=\sum_{m=-j-1}^{j}\frac{(-1)^m q^{(\mu+1)m^2+\mu m}}{(q;q)_{j-m}(q;q)_{j+m+1}}.$$

Following the proof of (1.8) using (2.10) and (4.1), we arrive at

$$\begin{aligned}\xi^*(k,\nu,n)=&\sum_{s_k=0}^{n}\frac{q^{s_k^2}}{(q;q)_{n-s_k}}\sum_{s_{k-1}=0}^{s_k}\frac{q^{s_{k-1}^2}}{(q;q)_{n-s_{k-1}}}\cdots\sum_{s_{\nu+1}=0}^{s_{\nu+2}}\frac{q^{s_{\nu+1}^2}}{(q;q)_{n-s_{\nu+1}}}\\&\sum_{s_\nu=0}^{s_{\nu+1}}\frac{q^{s_\nu^2+s_\nu}}{(q;q)_{s_{\nu+1}-s_\nu}}\cdots\sum_{s_1=0}^{s_2}\frac{q^{s_1^2+s_1}}{(q;q)_{s_2-s_1}}\chi^*(0,s_1).\end{aligned}$$

The proof of (1.9) is complete once we prove that

$$\chi^*(0,s_1)=\sum_{m=-s_1-1}^{s_1}\frac{(-1)^m q^{m^2}}{(q;q)_{s_1-m}(q;q)_{s_1+m+1}}=\frac{1}{(q^2;q^2)_{s_1}}. \tag{5.7}$$

Identity (5.7) is an identity discovered by C.F. Gauss (see [16, Sections 6-9] or [12, (2.1), (2.4)]) and the proof of (1.9) is complete. □

6. Garvan's k-ranks

In this section, we first discuss a few immediate consequences of Theorem 1.3. Next, we state and prove two lemmas needed in the proof of Theorem 1.3 before proceeding with its proof.

Letting n tend to infinity and multiplying both sides by $(q;q)_\infty$, we arrive at

$$\frac{1}{(q;q)_\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^{((2k+1)m^2+(2\nu+1)m)/2}(1-c)}{(1-cq^m)}$$
$$=\sum_{s_k=0}^{\infty}\sum_{s_{k-1}=0}^{s_k}\cdots\sum_{s_1=0}^{s_2}\frac{q^{U(s_k,s_{k-1},\cdots,s_2,s_1)}V_\infty(q)}{(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(cq;q)_{s_1}(q/c;q)_{s_1+1}}, \tag{6.1}$$

where

$$V_\infty(q)=\begin{cases}1-q^{s_{\nu+1}+1}/c, & \text{if } 0\le\nu\le k-1;\\ 1, & \text{if } \nu=k.\end{cases}$$

We can easily see that the case $\nu=0$ in (6.1) is equivalent to Garvan's identity [14, (4.1)].

Next, for $0\le\nu\le k-1$, by expressing

$$\frac{V_\infty(q)}{(q/c;q)_{s_1+1}}=\frac{c^{s_1}(c-q^{s_{\nu+1}+1})}{\prod_{j=1}^{s_1+1}(c-q^j)}$$

on the right hand side of (6.1) and then letting $c\to 0$, we see that the terms on the right hand side are zero except when $s_1=0$. The identity that remains is then equivalent to (1.5). Thus, (6.1) is a one variable generalization of (1.5). By the same argument, (1.12) for $0\le\nu\le k-1$, is readily seen as a one variable generalization of (1.8).

Finally, by setting $c=1$ in (1.12), we see that the terms on the left hand side vanishes except for the term when $m=0$. Thus we arrive at

$$\frac{1}{(q;q)_n^2}=\sum_{s_k=0}^{n}\sum_{s_{k-1}=0}^{s_k}\cdots\sum_{s_1=0}^{s_2}\frac{q^{U(s_k,s_{k-1},\cdots,s_2,s_1)}V_1(q)}{(q;q)_{n-s_k}(q;q)_{s_k-s_{k-1}}\cdots(q;q)_{s_2-s_1}(q;q)_{s_1}(q;q)_{s_1+1}},$$

where

$$V_1(q)=\begin{cases}1-q^{s_{\nu+1}+1}, & \text{if } 0\le\nu\le k-1;\\ 1-q^{n+1}, & \text{if } \nu=k.\end{cases}$$

Before proving Theorem 1.3, we state here two lemmas needed in our proof.

Lemma 6.1. *Let ℓ be a positive integer, let x be a non-zero complex number, and let q and c be any complex numbers. Then*

$$\sum_{m=-n}^{n} \frac{x^m q^{\ell(m^2+m)/2}}{(q;q)_{n-m}(q;q)_{n+m}(1-cq^m)} = \sum_{s=0}^{n} \frac{q^{s^2}}{(q;q)_{n-s}} \sum_{j=-s}^{s} \frac{x^j q^{((\ell-2)j^2+\ell j)/2}}{(q;q)_{s-j}(q;q)_{s+j}(1-cq^j)} \tag{6.2}$$

and

$$\sum_{m=-n-1}^{n} \frac{x^m q^{\ell(m^2+m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}(1-cq^m)} = \sum_{s=0}^{n} \frac{q^{s^2+s}}{(q;q)_{n-s}} \sum_{j=-s-1}^{s} \frac{x^j q^{(\ell-2)(j^2+j)/2}}{(q;q)_{s-j}(q;q)_{s+j+1}(1-cq^j)}. \tag{6.3}$$

Lemma 6.2. *Let ℓ be a positive integer and let q and c be any complex numbers. Then*

$$\sum_{m=-n}^{n} \frac{(-1)^m q^{\ell(m^2+m)/2}}{(q;q)_{n-m}(q;q)_{n+m}(1-cq^m)} = \left(1-q^{n+1}/c\right) \sum_{m=-n-1}^{n} \frac{(-1)^m q^{\ell(m^2+m)/2}}{(q;q)_{n-m}(q;q)_{n+m+1}(1-cq^m)}. \tag{6.4}$$

Except for the additional factor $1-cq^m$ in the denominator, the proofs of (6.2), (6.3), and (6.4) follow in exactly the same way as Bressoud's proof of (2.10) in [10, (18)], the proof of (4.1), and the proof of (4.14), respectively. As such, we omit proving Lemmas 6.1 and 6.2, and leave them to the interested reader.

Proof of Theorem 1.3. The proof is similar to that of (1.8).

We introduce some functions to simplify the presentation of our proof. Let

$$\xi_c(\mu,\nu,j) = \sum_{m=-j}^{j} \frac{(-1)^m q^{((2\mu+1)m^2+(2\nu+1)m)/2}(1-c)}{(q;q)_{j-m}(q;q)_{j+m}(1-cq^m)}$$

and

$$\chi_c(\mu,j) = \sum_{m=-j-1}^{j} \frac{(-1)^m q^{(2\mu+1)(m^2+m)/2}(1-c)}{(q;q)_{j-m}(q;q)_{j+m+1}(1-cq^m)}.$$

With the above notations, we may deduce from (6.2) and (6.3) that

$$\xi_c(\mu,\nu,n) = \sum_{m=0}^{n} \frac{q^{m^2}}{(q;q)_{n-m}} \xi_c(\mu-1,\nu,m) \tag{6.5}$$

and

$$\chi_c(\mu, n) = \sum_{m=0}^{n} \frac{q^{m^2+m}}{(q;q)_{n-m}} \chi_c(\mu-1, m). \tag{6.6}$$

Using the above notations, we may rewrite (6.4) as

$$\xi_c(\nu, \nu, n) = (1 - q^{n+1}/c)\chi_c(\nu, n). \tag{6.7}$$

Suppose $0 \le \nu \le k-1$. Then applying (6.5) repeatedly, we arrive at

$$\begin{aligned} \xi_c(k, \nu, n) = \sum_{s_k=0}^{n} \frac{q^{s_k^2}}{(q;q)_{n-s_k}} \sum_{s_{k-1}=0}^{s_k} \frac{q^{s_{k-1}^2}}{(q;q)_{n-s_{k-1}}} \\ \cdots \sum_{s_{\nu+2}=0}^{s_{\nu+3}} \frac{q^{s_{\nu+2}^2}}{(q;q)_{n-s_{\nu+2}}} \xi_c(\nu+1, \nu+1, s_{\nu+1}). \end{aligned} \tag{6.8}$$

Once we encounter $\xi_c(\nu+1, \nu+1, s_{\nu+1})$, we use (6.7) to replace $\xi_c(\nu+1, \nu+1, s_{\nu+1})$ with $(1 - q^{s_{\nu+1}+1}/c)\chi_c(\nu+1, s_{\nu+1})$. We then apply (6.6) to simplify the right hand side of (6.8) as

$$\begin{aligned} \xi(k, \nu, n) = \sum_{s_k=0}^{n} \frac{q^{s_k^2}}{(q;q)_{n-s_k}} \sum_{s_{k-1}=0}^{s_k} \frac{q^{s_{k-1}^2}}{(q;q)_{n-s_{k-1}}} \cdots \sum_{s_{\nu+2}=0}^{s_{\nu+3}} \frac{q^{s_{\nu+2}^2}}{(q;q)_{n-s_{\nu+2}}} \\ (1 + q^{s_{\nu+1}+1}/c) \sum_{s_{\nu+1}=0}^{s_{\nu+2}} \frac{q^{s_{\nu+1}^2+s_{\nu+1}}}{(q;q)_{s_{\nu+2}-s_{\nu+1}}} \cdots \sum_{s_1=0}^{s_2} \frac{q^{s_1^2+s_1}}{(q;q)_{s_2-s_1}} \chi_c(0, s_1). \end{aligned} \tag{6.9}$$

Using (6.7), we see that

$$\chi_c(0, s_1) = \frac{1}{(1 - q^{s_1+1}/c)} \xi_c(0, s_1) = \frac{1}{(cq;q)_{s_1}(q/c;q)_{s_1+1}} \tag{6.10}$$

by (1.11). Substituting (6.10) into (6.9) completes the proof of (1.12) for the cases $0 \le \nu \le k-1$.

For the case $\nu = k$, our starting point is the sum $\xi_c(k, k, n)$. Consequently, we skip the procedure involving the application of (6.5) and instead begin directly with the application of (6.7). The subsequent steps of the procedure remain unchanged. □

7. Summary

In this article, we accomplished several key objectives. We began by discussing and proving the finite form of Euler's Pentagonal Number Theorem (1.2). Next, we provided modified proofs of the two Rogers-Ramanujan identities, (1.3) and (1.4), using the Bailey transform (Theorem 3.1), while bypassing the need for the Bailey Lemma. We then established the necessary identities for proving the Andrews-Gordon identities, which are analogous to those previously discovered by Bressoud. Finally, we presented our proofs of the finite versions of the Andrews-Gordon identity, Bressoud's identity, and Garvan's generalization of Dyson's rank.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT-3.5 in order to improve English proficiency, refine sentence structures, and proofread various sections of the article. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Song Heng Chan reports financial support was provided by Government of Singapore Ministry of Education (Academic Research Fund, Tier 1 (RG15/23)). If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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